

SLIPPAGE TESTS FOR A SET OF GAMMA-VARIATES

BY

R. DOORNBOS AND H. J. PRINS ¹⁾

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1. Summary

In this paper a generalization is given of the significance tests for the largest and the smallest respectively of a set of estimated normal variances as suggested by W. G. COCHRAN (1941) and one of the present authors (cf. R. DOORNBOS (1956)) respectively. These tests only deal with the case where the samples from which the variances are estimated all have the same size.

The present paper gives a treatment which is also valid for different sample sizes. Further we consider the power function of the tests with respect to the alternative hypothesis that one of the variances has slipped to the right or, in the case of the test for the smallest variance, to the left.

2. Introduction and description of the tests

Suppose we have a set of random variables

$$(2.1) \quad \mathbf{u}_1, \dots, \mathbf{u}_k^{(2)}$$

distributed independently of one another according to gamma distributions with parameters $\alpha_1, \beta_1; \dots; \alpha_k, \beta_k$ respectively; that is to say the density function of $\mathbf{u}_i$ is

$$(2.2) \quad f(u_i) = \frac{1}{\Gamma(\alpha_i) \beta_i^{\alpha_i}} u_i^{\alpha_i-1} e^{-u_i/\beta_i}, \quad 0 \leq u_i \leq \infty,$$

where α_i and β_i are real positive numbers. As is well known the distribution of $\mathbf{t} = \chi^2 \sigma^2$, where χ^2 is distributed as a chi-square with ν degrees of freedom, is a special case of a gamma distribution, with parameters $\alpha = \nu/2$ and $\beta = 2\sigma^2$.

Now our problem is to find tests for the hypothesis

$$(2.3) \quad H_0: \beta_1 = \dots = \beta_k = \beta, \text{ say,}$$

against the alternatives

$$(2.4) \quad \left\{ \begin{array}{l} H_1: \beta_1 = \dots = \beta_{i-1} = \beta_{i+1} = \dots = \beta_k = \beta, \\ \beta_i = C_i \beta, C_i > 1, \end{array} \right.$$

¹⁾ Report SP 49 of the Statistical Department of the Mathematical Centre.

²⁾ Random variables are denoted by symbols printed in bold type.

for one unknown value of i and

$$(2.5) \quad \begin{cases} H_2: \beta_1 = \dots = \beta_{i-1} = \beta_{i+1} = \dots = \beta_k = \beta, \\ \beta_i = c_i \beta, \quad 0 < c_i < 1, \end{cases}$$

for one unknown value of i .

For both tests we compute the ratios

$$(2.6) \quad x_j = \frac{u_j}{\sum u_i}, \quad (j = 1, \dots, k).$$

Then, if we are testing H_0 against H_1 , the following incomplete B -integrals are determined:

$$(2.7) \quad \begin{cases} d_j = \frac{1}{B(\alpha_j, A - \alpha_j)} \int_{x_j}^1 x^{\alpha_j-1} (1-x)^{A-\alpha_j-1} dx \\ = 1 - I_{x_j}(\alpha_j, A - \alpha_j), \quad (j = 1, \dots, k), \end{cases}$$

where $A = \sum \alpha_i$. Next we define the test statistic d by

$$(2.8) \quad d = \min d_j.$$

If we reject H_0 when d takes a value $d \leq \varepsilon/k$, the level of significance lies between ε and $\varepsilon - \varepsilon^2/2$ as will be shown in section 4.

If all α_i have the same value, d corresponds to the smallest ratio x and our test reduces to COCHRAN's test.

Testing H_0 against H_2 requires computation of the integrals

$$(2.9) \quad \begin{cases} e_j = \frac{1}{B(\alpha_j, A - \alpha_j)} \int_0^{x_j} x^{\alpha_j-1} (1-x)^{A-\alpha_j-1} dx = 1 - d_j \\ = I_{x_j}(\alpha_j, A - \alpha_j). \end{cases}$$

We reject H_0 if

$$(2.10) \quad e = \min e_j \leq \varepsilon/k.$$

The level of significance is again a number between ε and $\varepsilon - \varepsilon^2/2$.

3. An optimum property of the tests if $\alpha_1 = \dots = \alpha_k$

D. R. TRUAX (1953) proved an optimum property of COCHRAN's test. In exactly the same way one can prove that our tests are optimal in the following sense if $\alpha_1 = \dots = \alpha_k$. Let D_0 be the decision that H_0 is true and let D_1 be the decision that H_0 is false and that $\beta_j = \max(\beta_1, \dots, \beta_k)$. Then, if $d = d_m$, i.e. if d_m is the smallest of $d_1, \dots, d_k$ the procedure

$$(3.1) \quad \begin{cases} \text{if } d \leq L_\varepsilon \text{ select } D_{1m}, \\ \text{if } d > L_\varepsilon \text{ select } D_0, \end{cases}$$

where L_ε is a constant determined by the condition that D_0 should be selected with probability $1-\varepsilon$ if H_0 is true, maximizes the probability of making the correct decision if the hypothesis H_1 is true.

When the hypothesis H_2 is true the analogous optimum property holds for our second test. In both cases ε/k is an approximation of the critical value of $\mathbf{d}$ and $\mathbf{e}$.

4. Proofs of the results stated in 2

To obtain the joint distribution of $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ as given by (2.6) and of $\mathbf{U} = \mathbf{u}_1 + \dots + \mathbf{u}_k$ we put

$$(4.1) \quad \begin{cases} u_1 = x_1 U, \\ \vdots \\ u_{k-1} = x_{k-1} U, \\ u_k = U(1 - x_1 - \dots - x_{k-1}). \end{cases}$$

The Jacobian of this transformation becomes U^{k-1} and the joint distribution of $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ and $\mathbf{U}$ is found to be given by the density function

$$(4.2) \quad \begin{cases} g(x_1, \dots, x_{k-1}, U) = \\ = \frac{\Gamma(A)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} x_1^{\alpha_1-1} \dots x_{k-1}^{\alpha_{k-1}-1} (1 - x_1 - \dots - x_{k-1})^{\alpha_k-1} \frac{U^{A-1} e^{-U/\beta}}{\Gamma(A) \beta^A}, \end{cases}$$

where $A = \alpha_1 + \dots + \alpha_k$.

Thus we see, as is well known that $\mathbf{U}$ has also a gamma distribution, with parameters $\alpha_1 + \dots + \alpha_k = A$ and β and moreover that the joint distribution of $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ is given by

$$(4.3) \quad h(x_1, \dots, x_{k-1}) = \frac{\Gamma(A)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} x_1^{\alpha_1-1} \dots x_{k-1}^{\alpha_{k-1}-1} (1 - x_1 - \dots - x_{k-1})^{\alpha_k-1}.$$

But the same derivation gives us the general result

$$(4.4) \quad \begin{cases} p(x_1, \dots, x_i) = \\ = \frac{\Gamma(A)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_i) \Gamma(\alpha_{i+1} + \dots + \alpha_k)} x_1^{\alpha_1-1} \dots x_i^{\alpha_i-1} (1 - x_1 - \dots - x_i)^{\alpha_{i+1} + \dots + \alpha_k - 1}, \\ \quad (i = 1, \dots, k-1), \end{cases}$$

if we consider instead of $\mathbf{u}_1, \dots, \mathbf{u}_k$ the $i+1$ variables $\mathbf{u}_1, \dots, \mathbf{u}_i$ and $\mathbf{u}_{i+1} + \dots + \mathbf{u}_k$ which are also independent of one another and which have gamma distributions with parameters $\alpha_1, \dots, \alpha_i, \alpha_{i+1} + \dots + \alpha_k$ and β .

We consider now a set of k real numbers $g_1, \dots, g_k$ ($0 \leq g_i \leq 1$) and the probabilities defined by

$$(4.5) \quad \begin{cases} p_i = P[\mathbf{x}_i \leq g_i], \\ p_{i,j} = P[\mathbf{x}_i \leq g_i \text{ and } \mathbf{x}_j \leq g_j], \quad (i \neq j) \\ \vdots \\ q_i = P[\mathbf{x}_i > g_i] \\ q_{i,j} = P[\mathbf{x}_i > g_i \text{ and } \mathbf{x}_j > g_j], \quad (i \neq j) \\ \vdots \end{cases}$$

all computed under H_0 . If we denote by P the probability that at least one of the ratios $\mathbf{x}_i$ does not exceed the corresponding value g_i , we have

$$(4.6) \quad P = \Sigma p_i - \Sigma p_{i,j} + \Sigma p_{i,j,l} \dots + (-1)^{k-1} p_{1,2,\dots,k},$$

where the r th summation is extended over all p 's with r subscripts; hence the r th sum has $\binom{k}{r}$ terms (cf. M. FRÉCHET (1940), or W. FELLER (1950), chapter 4).

For Q , the probability that at least one of the $\mathbf{x}_i$ exceeds g_i , we have

$$(4.7) \quad Q = \Sigma q_i - \Sigma q_{i,j} + \Sigma q_{i,j,l} \dots + (-1)^{k-1} q_{1,2,\dots,k}.$$

It follows from BONFERRONI's inequality (cf. M. FRÉCHET (1940), or W. FELLER (1950), chapter 4) that

$$(4.8) \quad \Sigma p_i - \Sigma p_{i,j} \leq P \leq \Sigma p_i$$

and

$$(4.9) \quad \Sigma q_i - \Sigma q_{i,j} \leq Q \leq \Sigma q_i.$$

In the latter part of this section we shall prove the inequalities

$$(4.10) \quad p_{i,j} \leq p_i p_j$$

and

$$(4.11) \quad q_{i,j} \leq q_i q_j.$$

If we now determine the numbers $g_{i,\varepsilon}$ so that all p_i are equal to ε/k , or according to (4.4)

$$(4.12) \quad P[\mathbf{x}_i \leq g_{i,\varepsilon}] = I_{g_{i,\varepsilon}}(\alpha_i, A - \alpha_i) = \varepsilon/k,$$

then we get from (4.8) and (4.10)

$$\Sigma p_i - \Sigma p_i p_j \leq \Sigma p_i - \Sigma p_{i,j} \leq P_\varepsilon \leq \Sigma p_i,$$

or

$$\varepsilon - (k-1)\varepsilon^2/2k \leq P_\varepsilon \leq \varepsilon,$$

or for $k \geq 2$

$$(4.13) \quad \varepsilon - \varepsilon^2/2 \leq P_\varepsilon \leq \varepsilon,$$

if P_ε is defined by

$$(4.14) \quad P_\varepsilon = P[\min(\mathbf{x}_i - g_{i,\varepsilon}) \leq 0].$$

In the same way we get

$$(4.15) \quad \varepsilon - \varepsilon^2/2 \leq Q_\varepsilon \leq \varepsilon,$$

if the numbers $G_{i,\varepsilon}$ are determined so as to make all q_i equal to ε/k and if Q_ε is the probability that at least one of the ratios $\mathbf{x}_i$ exceeds the corresponding value $G_{i,\varepsilon}$.

As the procedure described in section 2 to test H_0 against the sets of alternatives H_2 and H_1 respectively gives us the probabilities P_ε and

Q_e respectively of rejecting H_0 when H_0 is true, these probabilities lie between the bounds stated there.

We now proceed to prove the inequalities (4.10) and (4.11). First it is shown that (4.10) and (4.11) are equivalent. We have

$$p_i = 1 - q_i \text{ and } p_j = 1 - q_j$$

and consequently

$$(4.16) \quad p_i(1 - p_j) = q_j(1 - q_i).$$

Further

$$(4.17) \quad p_i - p_{i,j} = q_j - q_{i,j} (= P[\mathbf{x}_i \leq g_j \text{ and } \mathbf{x}_j > g_i]).$$

From (4.16) and (4.17) we obtain

$$(4.18) \quad p_i p_j - p_{i,j} = q_i q_j - q_{i,j},$$

which proves the equivalence of (4.10) and (4.11). Thus it is sufficient to prove (4.10)¹⁾ and we need only consider values g_i and g_j such that $g_i + g_j \leq 1$, for when $g_i + g_j > 1$, $q_{i,j} = 0$ and so (4.11) and (4.10) are obviously true.

It is easily seen that (4.10) is equivalent with

$$(4.19) \quad \frac{p_{i,j}}{p_j} \leq \frac{p_i - p_{i,j}}{q_j},$$

or

$$(4.20) \quad P[\mathbf{x}_i \leq g_i | \mathbf{x}_j \leq g_j] \leq P[\mathbf{x}_i \leq g_i | \mathbf{x}_j > g_j].$$

From (4.4) it follows that the left hand member $L(g_i, g_j)$ of (4.20) equals

$$C \frac{\int_0^{g_j} \int_0^{g_i} x_j^{\alpha_j-1} x_i^{\alpha_i-1} (1 - x_i - x_j)^{A - \alpha_i - \alpha_j - 1} dx_i dx_j}{\int_0^{g_j} x_j^{\alpha_j-1} (1 - x_j)^{A - \alpha_j - 1} dx_j},$$

where

$$C = \frac{\Gamma(A - \alpha_j)}{\Gamma(\alpha_i) \Gamma(A - \alpha_i - \alpha_j)}.$$

Putting $x_i = v(1 - x_j)$ we get

$$(4.21) \quad \left\{ \begin{aligned} L(g_i, g_j) &= C \frac{\int_0^{g_j} \int_0^{g_i/(1-x_j)} v^{\alpha_i-1} (1-v)^{A-\alpha_i-\alpha_j-1} x_j^{\alpha_j-1} (1-x_j)^{A-\alpha_j-1} dv dx_j}{\int_0^{g_j} x_j^{\alpha_j-1} (1-x_j)^{A-\alpha_j-1} dx_j} \\ &\leq \int_0^{g_i/(1-g_j)} v^{\alpha_i-1} (1-v)^{A-\alpha_i-\alpha_j-1} dv. \end{aligned} \right.$$

¹⁾ The following proof, which is substantially simpler than another one which was developed by the authors, has been found by H. KESTEN, assistant of the Statistical Department, as a special case of the proof of the more general inequality

$$p_{i,j,\dots,l} \leq p_i p_j \dots p_l.$$

Similarly the right hand member $R(g_i, g_j)$ of (4.20) is found to be equal to

$$(4.22) \quad \left\{ C \frac{\int_{g_j}^1 \int_{g_i}^{\min(g_i, 1-x_j)} x_i^{\alpha_i-1} x_j^{\alpha_j-1} (1-x_i-x_j)^{A-\alpha_i-\alpha_j-1} dx_i dx_j}{\int_{g_j}^1 x_j^{\alpha_j-1} (1-x_j)^{A-\alpha_j-1} dx_j} \right. \\ \left. \geq C \int_0^{g_j/(1-g_j)} v^{\alpha_i-1} (1-v)^{A-\alpha_i-\alpha_j-1} dv. \right.$$

So it follows from (4.21) and (4.22) that (4.20) holds.

5. The power of the tests

In this section we shall derive upper and lower bounds for the probabilities of making a correct decision, following the procedure described in section 3, under the hypotheses H_1 and H_2 .

In the first case, i.e. when H_1 is true, we assume that β_i is the parameter which has slipped to the right, i.e. $\beta_i = C_i \beta$, $C_i > 1$. Then we prove that Q_i , the probability of making the correct decision lies between the limits

$$(5.1) \quad \{1 - I_{B_i}(\alpha_i, A - \alpha_i)\} (1 - \varepsilon) \leq Q_i \leq \{1 - I_{B_i}(\alpha_i, A - \alpha_i)\},$$

where

$$(5.2) \quad B_i = \frac{G_{i,\varepsilon}}{C_i - (C_i - 1) G_{i,\varepsilon}},$$

where $G_{i,\varepsilon}$ is determined so as to make

$$(5.3) \quad I_{G_{i,\varepsilon}}(\alpha_i, A - \alpha_i) = 1 - \varepsilon/k.$$

When C_i becomes large Q_i converges to the upper bound given by the right hand member of (5.1).

When H_2 is true and β_j has slipped to the left, i.e. $\beta_j = c_j \beta$, $0 \leq c_j < 1$ the following limits can be derived for P_j , the probability of making the correct decision in this case.

$$(5.4) \quad \{I_{b_j}(\alpha_j, A - \alpha_j)\} (1 - \varepsilon) \leq P_j \leq I_{b_j}(\alpha_j, A - \alpha_j),$$

where

$$(5.5) \quad b_j = \frac{g_{j,\varepsilon}}{c_j + (1 - c_j) g_{j,\varepsilon}}$$

and $g_{j,\varepsilon}$ is determined from

$$(5.6) \quad I_{g_{j,\varepsilon}}(\alpha_j, A - \alpha_j) = \varepsilon/k.$$

Again for small values of c_j

$$P_j \approx I_{b_j}(\alpha_j, A - \alpha_j).$$

In order to prove (5.1) we may assume without loss of generality that $i = 1$ and then we put $u_1/C_1 = v_1$, thus v_1 has a gamma distribution with

parameters α_1 and β . The probability Q_1 of making the correct decision is

$$\begin{aligned} Q_1 &= P[\mathbf{d}_1 = \min \mathbf{d}_j \text{ and } \mathbf{d}_1 < \varepsilon/k] \\ &\geq P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_2 > \varepsilon/k \dots \text{ and } \mathbf{d}_k > \varepsilon/k] \\ &= P[\mathbf{d}_1 < \varepsilon/k] - P[(\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_2 < \varepsilon/k); \dots; \text{ or } (\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_k < \varepsilon/k)]. \end{aligned}$$

Thus the following inequality holds

$$(5.7) \quad P[\mathbf{d}_1 < \varepsilon/k] - \sum_{j=2}^k P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_j < \varepsilon/k] \leq Q_1 \leq P[\mathbf{d}_1 < \varepsilon/k].$$

We have

$$\begin{aligned} P[\mathbf{d}_1 < \varepsilon/k] &= P[\mathbf{x}_1 > G_{1,\varepsilon}] \\ &= P\left[\frac{G_{1,\varepsilon} \mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k + (G_{1,\varepsilon} - 1) \mathbf{v}_1} > G_{1,\varepsilon}\right] \\ &= P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k} > \frac{G_{1,\varepsilon}}{G_{1,\varepsilon} - (G_{1,\varepsilon} - 1) G_{1,\varepsilon}}\right] \\ &= P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k} > B_1\right] \end{aligned}$$

(cf. (5.2)).

The distribution of $\mathbf{v}_1/(\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k)$ is the distribution of $\mathbf{x}_1$ under H_0 and therefore known. In fact

$$(5.8) \quad P[\mathbf{d}_1 < \varepsilon/k] = 1 - I_{B_1}(\alpha_1, A - \alpha_1).$$

Further we have

$$(5.9) \quad \left\{ \begin{aligned} &P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_j < \varepsilon/k] \\ &= P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k} > B_1 \text{ and } \frac{\mathbf{u}_j}{\mathbf{v}_1 + \dots + \mathbf{u}_k + (G_{1,\varepsilon} - 1) \mathbf{v}_1} > G_{1,\varepsilon}\right] \\ &\leq P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k} > B_1 \text{ and } \frac{\mathbf{u}_j}{\mathbf{v}_1 + \dots + \mathbf{u}_k} > G_{1,\varepsilon}\right] \leq \\ &\quad (\text{according to (4.11)}) \\ &\leq P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_k} > B_1\right] \cdot P\left[\frac{\mathbf{u}_j}{\mathbf{v}_1 + \dots + \mathbf{u}_k} > G_{j,\varepsilon}\right] \\ &= [1 - I_{B_1}(\alpha_1, A - \alpha_1)] \varepsilon/k. \end{aligned} \right.$$

Substituting (5.8) and (5.9) into (5.7) we get

$$(5.10) \quad \left\{ \begin{aligned} &[1 - I_{B_1}(\alpha_1, A - \alpha_1)] (1 - \varepsilon) \leq [1 - I_{B_1}(\alpha_1, A - \alpha_1)] (1 - (k-1) \varepsilon/k) \\ &\leq Q_1 \leq [1 - I_{B_1}(\alpha_1, A - \alpha_1)], \end{aligned} \right.$$

which proves (5.1). When C_1 is large $P[\mathbf{d}_j < \varepsilon/k]$ will for $j \neq 1$ be much smaller than ε/k and therefore in that case Q_1 converges to its upper bound.

The inequalities (5.4) can be derived in the same way.

6. Tables and nomograms

To obtain the values d_j and e_j as defined by (2.7) and (2.9) and to evaluate the power functions (5.4) and (5.5) we need suitable tables or nomograms of the incomplete B function.

When all α_i are equal, the smallest d_j' corresponds to the largest ratio x_j and the smallest e_i corresponds to the smallest ratio x_i . Further the critical values $G_{i,\varepsilon}$ of x_i when testing the largest ratio and $g_{i,\varepsilon}$ for testing the smallest ratio are then all equal:

$$\begin{aligned} G_{i,\varepsilon} &= G_\varepsilon \quad (i = 1, \dots, k), \\ g_{i,\varepsilon} &= g_\varepsilon \quad (i = 1, \dots, k). \end{aligned}$$

Therefore in this case it suffices to have tables with these critical values with entries k and the common parameter value α . These tables may be found in C. EISENHART, M. W. HASTAY and W. A. WALLIS (1947) ($\varepsilon=0.05$ and 0.01) for the first test and in R. DOORNBOS (1956) ($\varepsilon=0.05$) for the second one.

When unequal values among the α_i occur the minimum d value may be found in most cases by means of PEARSON's tables of the incomplete function (K. PEARSON (1934)).

The smallest e value, however, will, when it lies in the neighbourhood of ε/k and k is not very small, correspond to such a small ratio x that PEARSON's tables are not suitable for our purpose. In this case and also if the parameter values $A - \alpha_j$ are larger than 50, the nomograms of H. O. HARTLEY and E. R. FITCH (1951) may be used to obtain an approximation to the e -values.

To demonstrate the use of these charts we consider the following, fictitious, example. Suppose we have a group of ten machines turning out the same product, and we measure some property t on each specimen. Suppose we are interested in finding out whether one of the machines produces the product more regularly than the other ones do.

From the i th machine we have n_i observations t_{ij} ($j=1, \dots, n_i$). For each machine the sum u_i of squared deviations from the mean value is computed:

$$u_i = \sum_{j=1}^{n_i} (t_{ij} - t_{i.})^2,$$

where

$$t_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} t_{ij}.$$

Assuming that the t_{ij} are independent observations from a normal distribution $N(\mu_i, \sigma_i)$, u_i is an observation from a gamma distribution with parameters $\alpha_i = (n_i - 1)/2$ and $\beta_i = 2\sigma_i^2$.

Fictitious u -values are given in table 6.1.

Clearly machine no 5 will give the smallest e value. So we must evaluate

$$e = I_{0.028}(7, 63) = 1 - I_{0.972}(63, 7).$$

From the chart we find that $I_{0.972}(63, 7) \approx 0.996$. Thus e is smaller than $0.05/10$ and therefore we can reject the hypothesis that all σ_i are equal

TABLE 6.1

Machine	1	2	3	4	5	6	7	8	9	10	total
Number of observations	10	15	21	23	15	11	31	15	3	6	150
α_i	4.5	7	10	11	7	5	15	7	1	2.5	70
u_i	45.9	109.6	112.8	142.0	25.7	123.0	182.0	106.4	12.8	46.5	906.7
$x = \frac{u_j}{\sum u_j}$	0.051	0.121	0.124	0.157	0.028	0.136	0.201	0.117	0.014	0.051	1.000

at the level of significance 0.05 and it may be concluded that the fifth machine is the most accurate one.

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¹⁾ This chart has also been included in *Biometrika Tables for Statisticians*, Vol. I (Cambridge 1954).

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